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An Integral Equation Formulation of the Equations of Motion of an Incompressible Fluid

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PREFACE

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A set of coupled integral equations is derived from the incompressible Navier-Stokes equations and the continuity equation. These equations are based on a vorticity-velocity-enthalpy formulation and are exact. The equations consist of a generalization of the Biot-Savart law for determining the velocity, an integral expression of the momentum equation for determining the vorticity, and a boundary integral equation for determining the stagnation enthalpy. The equations are linear in each independent variable, with the nonlinearities entering only through the cross terms of the vorticity and velocity. They possess a number of interesting properties, including the total absence of spatial derivatives and the fact that the stagnation enthalpy, or pressure, is required only on the boundary of the fluid domain. In addition, since the vorticity is present in all volume integrals, the domain of integration in this case is restricted to the region of nonzero vorticity. All boundary conditions, and in particular the farfield boundary condition, are naturally incorporated in the formulation.

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LIST OF SYMBOLS

B	Specific stagnation enthalpy
G	Scalar Green's function
$\mathbf{G}$	Vector Green's function
H	Scalar regular function
$\mathbf{H}$	Vector regular function
$\mathbf{n}$	Surface normal vector, into fluid
p	Pressure
$\mathbf{r}$	Displacement vector ($\mathbf{x}$ - $\boldsymbol{\xi}$)
r	Displacement magnitude $|\mathbf{x} - \boldsymbol{\xi}|$
S	Boundary of fluid domain
$\mathbf{u}$	Velocity vector (bounded fluid domains)
$\mathbf{u}$	Disturbance velocity vector (unbounded fluid domains)
$\mathbf{U}_\infty$	Freestream velocity vector
V	Fluid domain
V^c	Complement of fluid domain
$\mathbf{x}$	Field point position vector
μ	Absolute fluid viscosity
ν	Kinematic fluid viscosity
$\boldsymbol{\xi}$	Source point position vector
ρ	Fluid density
$\boldsymbol{\omega}$	Vorticity vector

AN INTEGRAL EQUATION FORMULATION OF THE EQUATIONS OF MOTION OF AN INCOMPRESSIBLE FLUID

INTRODUCTION

In the course of various investigations it became apparent that it might be possible to express the equations of motion of an incompressible fluid solely in terms of integral equations. In fact, it turns out to be possible to derive such a set of coupled integral equations in what may be called the vorticity-velocity-enthalpy formulation. This report contains a derivation of these equations and a discussion of their properties.

The formulation employed here is similar to that used by Howe (reference 1) in his investigations of acoustic wave equations. In that work Howe employed a formulation in which the acoustic pressure was replaced by a stagnation enthalpy that included the dynamic pressure term. The present formulation generalizes the incompressible form of the stagnation enthalpy integral equation to include the viscous term and formulates the rest of the equations of motion in terms of integral equations as well. Various mathematical manipulations are then carried out on these equations to render them in their final form.

MATHEMATICAL FORMULATION

Consider the flow of an incompressible fluid in both bounded and unbounded domains. The velocity may then be expressed as the sum

$$\mathbf{U}_\infty + \mathbf{u}, \tag{1}$$

where in an unbounded fluid domain $\mathbf{u}$ is the disturbance velocity and $\mathbf{U}_\infty$ is the freestream velocity ($\mathbf{U}_\infty = \mathbf{U}_\infty(t)$), and in a bounded fluid domain $\mathbf{u}$ is the total velocity and $\mathbf{U}_\infty$ may be set to zero (see figures 1 and 2 for illustrations of bounded and unbounded domains). The governing differential equations are then, the continuity equation

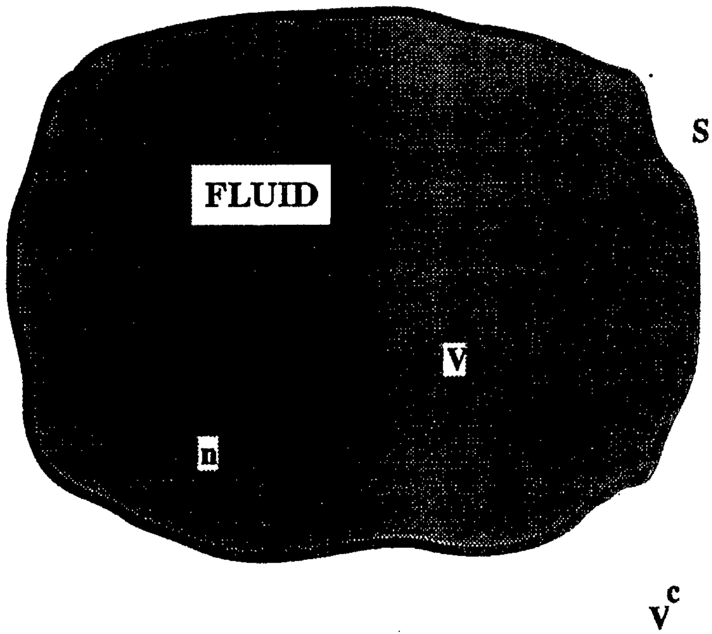

Figure 1. Bounded Fluid Domain

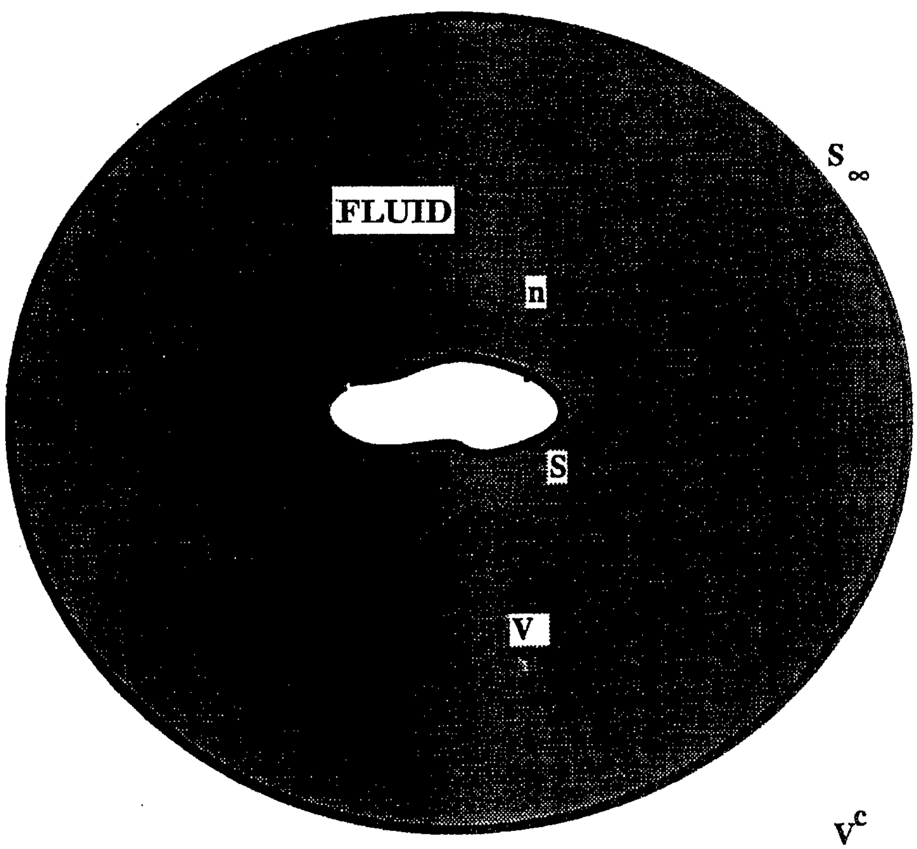

Figure 2. Unbounded Fluid Domain with the "Surface at Infinity"

$$\nabla \cdot \mathbf{u} = 0, \tag{2}$$

and the Navier-Stokes equations

$$\frac{\partial(\mathbf{U}_\infty+\mathbf{u})}{\partial t} + (\mathbf{U}_\infty+\mathbf{u})\cdot\nabla\,\mathbf{u} = -\frac{\nabla p}{\rho} + \nu\nabla^2\mathbf{u}. \tag{3}$$

The Navier-Stokes equations may also be written as

$$\frac{\partial(\mathbf{U}_\infty+\mathbf{u})}{\partial t} + \nabla\mathbf{B} - (\mathbf{U}_\infty+\mathbf{u}) \times \boldsymbol{\omega} = -\nu\,\nabla\times\boldsymbol{\omega}, \tag{4}$$

where the specific stagnation enthalpy B is defined as

$$B = \frac{p - p_\infty}{\rho} + \frac{1}{2}\left[(\mathbf{U}_\infty+\mathbf{u})\cdot(\mathbf{U}_\infty+\mathbf{u}) - \mathbf{U}_\infty\cdot\mathbf{U}_\infty\right], \tag{5}$$

and

$$\boldsymbol{\omega} = \nabla\times\mathbf{u} \tag{6}$$

is the vorticity.

The usual conditions on the flow boundaries are the no-flux boundary condition

$$\mathbf{n}\cdot\mathbf{u} = \mathbf{n}\cdot\mathbf{u}_B \text{ on S}, \tag{7}$$

and the no-slip boundary condition

$$\mathbf{n}\times\mathbf{u} = \mathbf{n}\times\mathbf{u}_B \text{ on S}, \tag{8}$$

where S is the boundary of the fluid domain V, and $\mathbf{u}_B$ is the velocity of the boundary. For unbounded flows there is also a condition at infinity that may be expressed as

$$\mathbf{u} = O\left(\frac{1}{r}\right), \text{ as } r \rightarrow \infty. \tag{9}$$

The derivation of the integral equation formulation requires two integral identities. The first identity is the vector identity (reference 2):

$$\beta\mathbf{a} = -\oint\!\!\!\oint_S \left[(\mathbf{n}\cdot\mathbf{a})\mathbf{G} - \mathbf{G}\times(\mathbf{n}\times\mathbf{a})\right]dS - \iiint_V \left[\mathbf{G}\times(\nabla\times\mathbf{a}) - (\nabla\cdot\mathbf{a})\mathbf{G}\right]dV, \tag{10}$$

where

$$\beta = \begin{cases} 4\pi \text{ in V} \\ 2\pi \text{ on S,} \\ 0 \text{ in Vc} \end{cases} \tag{11}$$

and $\mathbf{G}$ is any vector Green's function of the form

$$\mathbf{G} = \frac{\mathbf{r}}{r3} + \mathbf{H}(\mathbf{r}), \tag{12}$$

where $\mathbf{H}(\mathbf{r})$ is a regular vector function and $\mathbf{r} = \mathbf{x} - \boldsymbol{\xi}$. This identity holds for any vector field that is differentiable and for which the integrals exist. For reasons that will become clear subsequently, it shall be assumed that the curl of $\mathbf{G}$ is always zero.

The second integral identity is a generalization of Green's third identity (reference 3):

$$\beta\phi = \iint_S \left(\frac{\partial\phi}{\partial n}G - \phi\frac{\partial G}{\partial n}\right)dS - \iiint_V \nabla^2\phi G dV, \tag{13}$$

where G is any scalar Green's function of the form

$$G = \frac{1}{r} + H(r), \tag{14}$$

and H(r) is a regular function. This identity holds for any scalar field that is differentiable and for which the integrals exist.

The integral equations will be derived for the case of a bounded flow domain. If an unbounded flow domain is under consideration, care must be taken when considering the contribution of the flow boundary "at infinity." In that case the integrals over the "surface at infinity" may be eliminated by assuming that the disturbance velocity and the vorticity vanish sufficiently fast at large distances and by defining the stagnation enthalpy as in equation (5) so that it, too, goes to zero at infinity.

From the integral identity (10) and the fact that

$$\nabla \cdot \boldsymbol{\omega} = \nabla \cdot (\nabla \times \mathbf{u}) = 0, \tag{15}$$

one immediately finds that

$$\beta \mathbf{u} = - \oint\!\!\!\oint_S \left[(\mathbf{n} \cdot \mathbf{u})\mathbf{G} - \mathbf{G} \times (\mathbf{n} \times \mathbf{u})\right] dS - \iiint_V \mathbf{G} \times \boldsymbol{\omega}\, dV, \tag{16}$$

which expresses the velocity field in terms of the vorticity in the fluid domain and a boundary contribution. This equation is merely a generalization of the Biot-Savart law. Similarly, equation (3) may be rewritten as

$$\nabla \times \boldsymbol{\omega} = \frac{1}{\nu}\left[(\mathbf{U}_\infty \times \mathbf{u}) \times \boldsymbol{\omega} - \frac{\partial(\mathbf{U}_\infty + \mathbf{u})}{\partial t} - \nabla B\right]. \tag{17}$$

Therefore, the integral identity (10) yields

$$\beta \boldsymbol{\omega} = - \oint\!\!\!\oint_S \left[\mathbf{G}(\mathbf{n} \cdot \boldsymbol{\omega}) - \mathbf{G} \times (\mathbf{n} \times \boldsymbol{\omega})\right] dS$$

$$+ \frac{1}{\nu} \iiint_V \mathbf{G} \times \left[\frac{\partial(\mathbf{U}_\infty + \mathbf{u})}{\partial t} - (\mathbf{U}_\infty + \mathbf{u}) \times \boldsymbol{\omega} + \nabla B\right] dV. \tag{18}$$

However, since

$$\nabla B \times \mathbf{G} = \nabla \times (B\mathbf{G}) - B\nabla \times \mathbf{G},$$
$$= \nabla \times (B\mathbf{G}), \tag{19}$$

where it has been assumed that the curl of the vector Green's function **G** is zero, one finds that the enthalpy is required only on the boundary of the fluid domain, and equation (18) becomes

$$\beta \boldsymbol{\omega} = - \oiint_S \left[\mathbf{G}(\mathbf{n} \cdot \boldsymbol{\omega}) - \mathbf{G} \times (\mathbf{n} \times \boldsymbol{\omega})\right] dS + \frac{1}{\nu} \oiint_S B\,\mathbf{G} \times \mathbf{n}\, dS$$
$$+ \frac{1}{\nu} \iiint_V \mathbf{G} \times \left[\frac{\partial(\mathbf{U}_\infty + \mathbf{u})}{\partial t} - (\mathbf{U}_\infty + \mathbf{u}) \times \boldsymbol{\omega}\right] dV. \tag{20}$$

Note that the time derivative may be expressed as a material time derivative, yielding

$$\beta \boldsymbol{\omega} = - \oiint_S \left[\mathbf{G}(\mathbf{n} \cdot \boldsymbol{\omega}) - \mathbf{G} \times (\mathbf{n} \times \boldsymbol{\omega})\right] dS + \frac{1}{\nu} \oiint_S B\,\mathbf{G} \times \mathbf{n}\, dS$$
$$+ \frac{1}{\nu} \iiint_V \mathbf{G} \times \left\{\frac{D(\mathbf{U}_\infty + \mathbf{u})}{Dt} - \frac{1}{2} \nabla\left[(\mathbf{U}_\infty + \mathbf{u}) \cdot (\mathbf{U}_\infty + \mathbf{u})\right]\right\} dV; \tag{21}$$

or, operating on the gradient term in a manner similar to that employed in equation (19), one finds

$$\beta \boldsymbol{\omega} = - \oiint_S \left[\mathbf{G}(\mathbf{n} \cdot \boldsymbol{\omega}) - \mathbf{G} \times (\mathbf{n} \times \boldsymbol{\omega})\right] dS$$
$$+ \frac{1}{\nu} \oiint_S \left[B - \frac{1}{2} (\mathbf{U}_\infty + \mathbf{u}) \cdot (\mathbf{U}_\infty + \mathbf{u})\right] \mathbf{G} \times \mathbf{n}\, dS$$
$$+ \frac{1}{\nu} \iiint_V \mathbf{G} \times \frac{D(\mathbf{U}_\infty + \mathbf{u})}{Dt}\, dV. \tag{22}$$

(See Hildebrand (reference 4) for the applicable vector identities.)

Finally, to obtain an equation for the specific stagnation enthalpy, note that if the divergence of equation (4) is taken, one is left with

$$\nabla^2 B = \nabla\cdot\left[(\mathbf{U}_\infty+\mathbf{u})\times\boldsymbol{\omega}\right]. \tag{23}$$

Similarly, equation (4) can be employed to show that on the boundary

$$\begin{aligned}\left.\frac{\partial B}{\partial n}\right|_S &= \mathbf{n}\cdot\left[-\frac{\partial(\mathbf{U}_\infty+\mathbf{u})}{\partial t}+(\mathbf{U}_\infty+\mathbf{u})\times\boldsymbol{\omega}+\nu\nabla^2\mathbf{u}\right],\\ &= \mathbf{n}\cdot\left[-\frac{\partial(\mathbf{U}_\infty+\mathbf{u})}{\partial t}+(\mathbf{U}_\infty+\mathbf{u})\times\boldsymbol{\omega}-\nu\nabla\times\boldsymbol{\omega}\right].\end{aligned} \tag{24}$$

Hence, employing integral identity (13) and using expressions (23) and (24) derived above, one finds that

$$\begin{aligned}\beta B+\oint\!\!\oint_S B\frac{\partial G}{\partial n}dS &= \oint\!\!\oint_S\left\{-\mathbf{n}\cdot\frac{\partial(\mathbf{U}_\infty+\mathbf{u})}{\partial t}+\mathbf{n}\cdot\left[(\mathbf{U}_\infty+\mathbf{u})\times\boldsymbol{\omega}\right]-\nu\mathbf{n}\cdot(\nabla\times\boldsymbol{\omega})\right\}G\,dS\\ &\quad-\iiint_V\nabla\cdot\left[(\mathbf{U}_\infty+\mathbf{u})\times\boldsymbol{\omega}\right]G\,dV.\end{aligned} \tag{25}$$

Employing the identities

$$\begin{aligned}(\nabla_{\boldsymbol{\xi}}\times\boldsymbol{\omega})G &= \nabla_{\boldsymbol{\xi}}\cdot(\boldsymbol{\omega}G)+\nabla G\times\boldsymbol{\omega},\\ \nabla_{\boldsymbol{\xi}}\cdot(\mathbf{u}\times\boldsymbol{\omega})G &= \nabla_{\boldsymbol{\xi}}\cdot\left[(\mathbf{u}\times\boldsymbol{\omega})G\right]+\nabla G\cdot(\mathbf{u}\times\boldsymbol{\omega}),\end{aligned} \tag{26}$$

and the theorems of Gauss and Stokes, one can find the third and fourth terms on the right-hand side of equation (25) yield

$$\begin{aligned}-\nu\oint\!\!\oint_S(\mathbf{n}\cdot\nabla\times\boldsymbol{\omega})G\,dS &= -\nu\left[\oint\!\!\oint_S\mathbf{n}\cdot\nabla\times(\boldsymbol{\omega}G)dS-\oint\!\!\oint_S\mathbf{n}\cdot(\nabla G\times\boldsymbol{\omega})dS\right],\\ &= -\nu\left[\oint_C\boldsymbol{\omega}G\cdot d\mathbf{S}-\oint\!\!\oint_S\mathbf{n}\cdot(\nabla G\times\boldsymbol{\omega})dS\right],\\ &= \nu\oint\!\!\oint_S\mathbf{n}\cdot(\nabla G\times\boldsymbol{\omega})dS,\end{aligned} \tag{27}$$

and

$$\iiint\limits_V \left\{\nabla\cdot\left[(\mathbf{U}_\infty+\mathbf{u})\times\boldsymbol{\omega}\right]\right\} G dV = \iiint\limits_V \nabla\cdot\left\{\left[(\mathbf{U}_\infty+\mathbf{u})\times\boldsymbol{\omega}\right]G\right\} dV$$

$$- \iiint\limits_V \nabla G\cdot\left[(\mathbf{U}_\infty+\mathbf{u})\times\boldsymbol{\omega}\right]dV,$$

$$= \oiint\limits_S \mathrm{n}\cdot\left\{\left[(\mathbf{U}_\infty+\mathbf{u})\times\boldsymbol{\omega}\right]G\right\}dS$$

$$- \iiint\limits_V \nabla G\cdot\left[(\mathbf{U}_\infty+\mathbf{u})\times\boldsymbol{\omega}\right]dV. \tag{28}$$

Therefore, equation (25) becomes

$$\beta B + \oiint\limits_S B\,\frac{\partial G}{\partial n}\,dS = \oiint\limits_S \left[-\mathbf{n}\cdot\frac{\partial(\mathbf{U}_\infty+\mathbf{u})}{\partial t}\,G + \nu\mathbf{n}\cdot(\nabla G\times\boldsymbol{\omega})\right]dS$$

$$+ \iiint\limits_V \nabla G\cdot\left[(\mathbf{U}_\infty+\mathbf{u})\times\boldsymbol{\omega}\right]dV, \tag{29}$$

or, since

$$\mathbf{n}\cdot(\nabla G\times\boldsymbol{\omega}) = -\nabla G\cdot(\mathbf{n}\times\boldsymbol{\omega}), \tag{30}$$

one finally arrives at the integral equation for the enthalpy:

$$\beta B + \oiint\limits_S B\,\frac{\partial G}{\partial n}\,dS = -\oiint\limits_S \left[\mathbf{n}\cdot\frac{\partial(\mathbf{U}_\infty+\mathbf{u})}{\partial t}\,G + \nu\nabla G\cdot(\mathbf{n}\times\boldsymbol{\omega})\right]dS$$

$$+ \iiint\limits_V \nabla G\cdot\left[(\mathbf{U}_\infty+\mathbf{u})\times\boldsymbol{\omega}\right]dV. \tag{31}$$

Therefore, the set of coupled integral equations in the vorticity-velocity-enthalpy formulation is, for the case of general boundary motion,

$$\beta \mathbf{u} = - \oiint_S \left[(\mathbf{n}\cdot\mathbf{u})\,\mathbf{G} - \mathbf{G}\times(\mathbf{n}\times\mathbf{u})\right] dS - \iiint_V \mathbf{G}\times\boldsymbol{\omega}\, dV, \tag{32}$$

and

$$\beta\boldsymbol{\omega} = - \oiint_S \left[\mathbf{G}(\mathbf{n}\cdot\boldsymbol{\omega}) - \mathbf{G}\times(\mathbf{n}\times\boldsymbol{\omega})\right] dS + \frac{1}{\nu}\oiint_S B\,(\mathbf{G}\times\mathbf{n})\, dS$$
$$+ \frac{1}{\nu}\iiint_V \mathbf{G}\times\left[\frac{\partial(\mathbf{U}_\infty+\mathbf{u})}{\partial t} - (\mathbf{U}_\infty+\mathbf{u})\times\boldsymbol{\omega}\right] dV, \tag{33}$$

and

$$\beta B + \oiint_S B\,\frac{\partial G}{\partial n}\, dS = -\oiint_S \left[\mathbf{n}\cdot\frac{\partial(\mathbf{U}_\infty+\mathbf{u})}{\partial t}\, G + \nu\nabla G\cdot(\mathbf{n}\times\boldsymbol{\omega})\right] dS$$
$$+ \iiint_V \nabla G\cdot\left[(\mathbf{U}_\infty+\mathbf{u})\times\boldsymbol{\omega}\right] dV. \tag{34}$$

If the free-space vector and scalar Green's functions

$$\mathbf{G} = \frac{\mathbf{r}}{r^3} \tag{35}$$

and

$$G = \frac{1}{r} \tag{36}$$

are employed, then the integrand of the time-derivative term in equation (33) may be written as

$$\frac{\mathbf{r}\times\frac{\partial}{\partial t}(\mathbf{U}_\infty+\mathbf{u})}{r^3} = \nabla\left(\frac{1}{r}\right)\times\frac{\partial}{\partial t}(\mathbf{U}_\infty+\mathbf{u}),$$
$$= \nabla\times\left[\left(\frac{1}{r}\right)\frac{\partial}{\partial t}(\mathbf{U}_\infty+\mathbf{u})\right] - \left(\frac{1}{r}\right)\frac{\partial\boldsymbol{\omega}}{\partial t}, \tag{37}$$

so that

$$\iiint_V \frac{\mathbf{r}\times\frac{\partial}{\partial t}(\mathbf{U}_\infty+\mathbf{u})}{r^3}\,dV = \oint\!\!\!\!\oint_S \frac{\mathbf{n}\times\frac{\partial}{\partial t}(\mathbf{U}_\infty+\mathbf{u})}{r}\,dS - \iiint_V \frac{\partial\boldsymbol{\omega}}{\partial t}\left(\frac{1}{r}\right)dV. \tag{38}$$

Therefore, with the Green's functions given by equations (35) and (36), the set of coupled integral equations (32)-(34) may be written as

$$\beta\mathbf{u} = -\oint\!\!\!\!\oint_S \left[\frac{(\mathbf{n}\cdot\mathbf{u})\mathbf{r}}{r^3} - \frac{\mathbf{r}\times(\mathbf{n}\times\mathbf{u})}{r^3}\right]dS - \iiint_V \frac{\mathbf{r}\times\boldsymbol{\omega}}{r^3}\,dV, \tag{39}$$

and

$$\beta\boldsymbol{\omega} = -\oint\!\!\!\!\oint_S \left[\frac{(\mathbf{n}\cdot\boldsymbol{\omega})\mathbf{r}}{r^3} - \frac{\mathbf{r}\times(\mathbf{n}\times\boldsymbol{\omega})}{r^3}\right]dS + \frac{1}{\nu}\oint\!\!\!\!\oint_S B\,\frac{(\mathbf{r}\times\mathbf{n})}{r^3}\,dS$$

$$+ \frac{1}{\nu}\oint\!\!\!\!\oint_S \frac{\mathbf{n}\times\frac{\partial}{\partial t}(\mathbf{U}_\infty+\mathbf{u})}{r}\,dS$$

$$- \frac{1}{\nu}\iiint_V \left\{\frac{\partial\boldsymbol{\omega}}{\partial t}\left(\frac{1}{r}\right) + \frac{\mathbf{r}\times[(\mathbf{U}_\infty+\mathbf{u})\times\boldsymbol{\omega}]}{r^3}\right\}dV, \tag{40}$$

and

$$\beta B + \oint\!\!\!\!\oint_S B\,\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS = -\oint\!\!\!\!\oint_S \left\{\mathbf{n}\cdot\frac{\partial(\mathbf{U}_\infty+\mathbf{u})}{\partial t}\left(\frac{1}{r}\right) + \nu\,\frac{\mathbf{r}\cdot(\mathbf{n}\times\boldsymbol{\omega})}{r^3}\right\}dS$$

$$+ \iiint_V \frac{\mathbf{r}\cdot[(\mathbf{U}_\infty+\mathbf{u})\times\boldsymbol{\omega}]}{r^3}\,dV. \tag{41}$$

In two dimensions, the free-field vector and scalar Green's functions are

$$\mathbf{G} = \frac{\mathbf{r}}{r^2}, \tag{42}$$

and

$$\mathbf{G} = \ln\left(\frac{1}{r}\right), \tag{43}$$

respectively. The corresponding integral equations in two dimensions are then

$$\beta \mathbf{u} = -\oint_S \left[\frac{(\mathbf{n}\cdot\mathbf{u})\mathbf{r}}{r^2} - \frac{\mathbf{r}\times(\mathbf{n}\times\mathbf{u})}{r^2}\right] dl - \iint_V \frac{\mathbf{r}\times\boldsymbol{\omega}}{r^2}\, dS, \tag{44}$$

and

$$\beta\boldsymbol{\omega} = \oint_S \frac{\mathbf{r}\times(\mathbf{n}\times\boldsymbol{\omega})}{r^2}\, dl + \frac{1}{\nu}\oint_S B\,\frac{\mathbf{r}\times\mathbf{n}}{r^2}\, dl + \frac{1}{\nu}\iint_V \left[\mathbf{n}\times\frac{\partial}{\partial t}(\mathbf{U}_\infty+\mathbf{u})\right]\ln\left(\frac{1}{r}\right) dS$$

$$- \frac{1}{\nu}\iint_V \left\{\frac{\partial\boldsymbol{\omega}}{\partial t}\ln\left(\frac{1}{r}\right) + \frac{\mathbf{r}\times\left[(\mathbf{U}_\infty+\mathbf{u})\times\boldsymbol{\omega}\right]}{r^2}\right\} dS, \tag{45}$$

and

$$\beta B + \oint_S B\frac{\partial}{\partial n}\ln\left(\frac{1}{r}\right) dl = -\oint_S \left\{\mathbf{n}\cdot\frac{\partial}{\partial t}(\mathbf{U}_\infty+\mathbf{u})\ln\left(\frac{1}{r}\right) + \nu\,\frac{\mathbf{r}\cdot(\mathbf{n}\times\boldsymbol{\omega})}{r^2}\right\} dl$$

$$+ \iint_V \frac{\mathbf{r}\cdot\left[(\mathbf{U}_\infty+\mathbf{u})\times\boldsymbol{\omega}\right]}{r^2}\, dS, \tag{46}$$

where, in two dimensions,

$$\beta = \begin{cases} 2\pi \text{ in V} \\ \pi \text{ on S}. \\ 0 \text{ in V}^c \end{cases} \tag{47}$$

DISCUSSION

The boundary conditions expressed in equations (7) and (8) may be readily implemented in equation (39) by substitution in the integral over the boundary, yielding

$$\beta\mathbf{u} = - \oint\!\!\oint_S \left[\frac{(\mathbf{n}\cdot\mathbf{u}_B)\mathbf{r}}{r^3} - \frac{\mathbf{r}\times(\mathbf{n}\times\mathbf{u}_B)}{r^3}\right] dS - \iiint_V \frac{\mathbf{r}\times\boldsymbol{\omega}}{r^3}\, dV. \tag{48}$$

Equation (40) also includes an integral over the boundary that involves the dot and cross products of the surface normal and the surface vorticity. The treatment of the boundary terms here is not as straightforward. It is shown in appendix C that, for rigid body motion with a no-slip condition, the dot product of the normal and the vorticity can be expressed in terms of the boundary condition; that is,

$$\mathbf{n}\cdot\boldsymbol{\omega} = -2\mathbf{n}\cdot\boldsymbol{\Omega}, \tag{49}$$

where the boundary condition has been decomposed into its translational $\mathbf{U}$ and rotational $\boldsymbol{\Omega}$ parts as

$$\mathbf{u}_B = \mathbf{U}+\mathbf{r}\times\boldsymbol{\Omega}. \tag{50}$$

However, the cross product of the normal and the vorticity cannot be expressed in terms of the boundary conditions since it involves derivatives of the velocity field normal to the boundary (see appendix C). Thus, for rigid body motion, equation (40) becomes

$$\begin{aligned}\beta\boldsymbol{\omega} = &- \oint\!\!\oint_S \left[\frac{\mathbf{r}(-2\mathbf{n}\cdot\boldsymbol{\Omega})}{r^3} - \frac{\mathbf{r}\times(\mathbf{n}\times\boldsymbol{\omega})}{r^3}\right] dS + \frac{1}{\nu}\oint\!\!\oint_S B\,\frac{(\mathbf{r}\times\mathbf{n})}{r^3}\, dS \\ &+ \frac{1}{\nu}\oint\!\!\oint_S \frac{\mathbf{n}\times\frac{\partial}{\partial t}(\mathbf{U}_\infty+\mathbf{u})}{r}\, dS \\ &- \frac{1}{\nu}\iiint_V \left\{ \frac{\partial\boldsymbol{\omega}}{\partial t}\left(\frac{1}{r}\right) + \frac{\mathbf{r}\times[(\mathbf{U}_\infty+\mathbf{u})\times\boldsymbol{\omega}]}{r^3}\right\} dV.\end{aligned} \tag{51}$$

Note that equation (41) involves the same cross product boundary term and therefore cannot be simplified by using the boundary conditions.

Equations (48), (51), and (41) now represent the system of equations to be solved both in the interior of the fluid and on its boundary. However, the nature of the equations changes depending on whether or not the boundary or interior is being considered. In the interior of the fluid the equations take the form

$$4\pi\mathbf{u} = - \oiint_S \left[\frac{(\mathbf{n}\cdot\mathbf{u}_B)\mathbf{r}}{r^3} - \frac{\mathbf{r}\times(\mathbf{n}\times\mathbf{u}_B)}{r^3}\right]dS - \iiint_V \frac{\mathbf{r}\times\boldsymbol{\omega}}{r^3}\,dV, \tag{52}$$

and

$$4\pi\boldsymbol{\omega} = - \oiint_S \left[\frac{\mathbf{r}(-2\mathbf{n}\cdot\boldsymbol{\Omega})}{r^3} - \frac{\mathbf{r}\times(\mathbf{n}\times\boldsymbol{\omega})}{r^3}\right]dS + \frac{1}{\nu}\oiint_S B\,\frac{(\mathbf{r}\times\mathbf{n})}{r^3}\,dS$$

$$+ \frac{1}{\nu}\oiint_S \frac{\mathbf{n}\times\frac{\partial}{\partial t}(\mathbf{U}_\infty+\mathbf{u})}{r}\,dS$$

$$- \frac{1}{\nu}\iiint_V \left\{\frac{\partial\boldsymbol{\omega}}{\partial t}\left(\frac{1}{r}\right) + \frac{\mathbf{r}\times\left[(\mathbf{U}_\infty+\mathbf{u})\times\boldsymbol{\omega}\right]}{r^3}\right\} dV, \tag{53}$$

where each equation has been written with its unknowns on the left-hand side. Note that no equation is needed for the stagnation enthalpy in the interior of the fluid since equation (53) requires the enthalpy only on the boundary.

If one writes the equations for the boundary values in the same form one finds

$$2\pi\boldsymbol{\omega} - \oiint_S \frac{\mathbf{r}\times(\mathbf{n}\times\boldsymbol{\omega})}{r^3}\,dS = - \oiint_S \frac{\mathbf{r}(-2\mathbf{n}\cdot\boldsymbol{\Omega})}{r^3}\,dS + \frac{1}{\nu}\oiint_S B\,\frac{(\mathbf{r}\times\mathbf{n})}{r^3}\,dS$$

$$+ \frac{1}{\nu}\oiint_S \frac{\mathbf{n}\times\frac{\partial}{\partial t}(\mathbf{U}_\infty+\mathbf{u})}{r}\,dS$$

$$- \frac{1}{\nu}\iiint_V \left\{\frac{\partial\boldsymbol{\omega}}{\partial t}\left(\frac{1}{r}\right) + \frac{\mathbf{r}\times\left[(\mathbf{U}_\infty+\mathbf{u})\times\boldsymbol{\omega}\right]}{r^3}\right\} dV, \tag{54}$$

and

$$2\pi B + \oiint_S B \frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS = - \oiint_S \left[\mathbf{n} \cdot \frac{\partial(\mathbf{U}_\infty + \mathbf{u})}{\partial t}\left(\frac{1}{r}\right) + \nu \frac{\mathbf{r} \cdot (\mathbf{n} \times \boldsymbol{\omega})}{r^3}\right] dS$$

$$+ \iiint_V \frac{\mathbf{r} \cdot [(\mathbf{U}_\infty + \mathbf{u}) \times \boldsymbol{\omega}]}{r^3} dV. \qquad (55)$$

Note that no equation is required for the velocities on the boundary since they are already specified by the boundary conditions.

Equations (52) through (55) represent an integral equation reformulation of the equations of motion of an incompressible fluid. These equations possess some interesting properties. First, they contain no spatial derivatives. Second, they only require knowledge of the "pressure" quantity (the specific stagnation enthalpy) on the boundary of the fluid domain. Third, if the variables $\mathbf{u}$, $\boldsymbol{\omega}$, and B are considered as independent, then these equations are linear in each variable, a condition one might call "pseudo-linear." Fourth, since all the volume integrals present in the equations contain the vorticity and since the farfield boundary condition is incorporated in the formulation, it is immediately apparent that the domain of interest in the fluid may be restricted solely to that region of the fluid in which the vorticity is nonzero. These facts naturally lead to speculation as to the usefulness that this formulation might have from a computational point of view. The fact that the equations contain no spatial derivatives suggests that they might not be prone to the requirement of artificial viscosity, which plagues so many finite-difference-based computational schemes. Similarly, since the integral equation formulation only requires knowledge of the "pressure" on the boundary, it may obviate the need for elaborate schemes to guarantee the convergence of the pressure calculation in the interior as is required in many differential approaches. The "pseudo-linearity" of the equations may allow the use of efficient iteration schemes for the solution of the equations. Finally, the ability to restrict the domain of consideration to only the domain of nonzero vorticity should, in typical high Reynolds number cases where the vorticity is exponentially small outside of a thin region, greatly reduce the number of nodes at which the solution must be obtained.

SUMMARY AND CONCLUSIONS

An integral formulation of the equations of motion of an incompressible fluid has been derived. The equations consist of a generalization of the Biot-Savart law for determining the velocity, an integral expression of the momentum equation for determining the vorticity, and a boundary integral equation for determining the stagnation enthalpy. The equations are linear in each independent variable, with the nonlinearities entering only through cross terms of the vorticity and velocity.

This formulation possesses several salient features, including the total absence of spatial derivatives, the fact that the stagnation enthalpy, or pressure, is required only on the boundary of the fluid domain and the fact that, since the vorticity is present in all volume integrals, the domain of integration in this case is restricted to the region of nonzero vorticity. In addition, all boundary conditions, and in particular the farfield boundary condition, are naturally incorporated in the formulation.

APPENDIX A

DERIVATION OF VECTOR INTEGRAL IDENTITY

One may start by noting that the vector identities

$$\begin{aligned} \nabla\times(\mathbf{G}\times\mathbf{u}) &= (\mathbf{u}\cdot\nabla)\mathbf{G}-(\mathbf{G}\cdot\nabla)\mathbf{u}+\mathbf{G}(\nabla\cdot\mathbf{u})-\mathbf{u}(\nabla\cdot\mathbf{G}), \\ \nabla(\mathbf{G}\cdot\mathbf{u}) &= (\mathbf{G}\cdot\nabla)\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{G}+\mathbf{G}\times(\nabla\times\mathbf{u})+\mathbf{u}\times(\nabla\times\mathbf{G}) \end{aligned} \tag{A-1}$$

may be added with the result that

$$\begin{aligned} \nabla\times(\mathbf{G}\times\mathbf{u})+\nabla(\mathbf{G}\cdot\mathbf{u}) &= 2(\mathbf{u}\cdot\nabla)\mathbf{G}+\mathbf{G}(\nabla\cdot\mathbf{u})-\mathbf{u}(\nabla\cdot\mathbf{G}) \\ &\quad+\mathbf{G}\times(\nabla\times\mathbf{u})+\mathbf{u}\times(\nabla\times\mathbf{G}). \end{aligned} \tag{A-2}$$

If one integrates this expression over the volume of the domain and assumes that the functions under examination are sufficiently well behaved for the integrals to exist, then one may apply the curl and gradient theorems,

$$\begin{aligned} \iiint_V \nabla\times\mathbf{F}\,dV &= \oiint_S \mathbf{n}\times\mathbf{F}\,dS, \\ \iiint_V \nabla f\,dV &= \oiint_S f\,\mathbf{n}\,dS, \end{aligned} \tag{A-3}$$

to the volume integral of the left-hand side of the expression to find that

$$\iiint_V \left[\nabla\times(\mathbf{G}\times\mathbf{u})+\nabla(\mathbf{G}\cdot\mathbf{u})\right]dV = \oiint_S \left[\mathbf{n}\times(\mathbf{G}\times\mathbf{u})+(\mathbf{G}\cdot\mathbf{u})\mathbf{n}\right]dS. \tag{A-4}$$

However, it can readily be shown that

$$\begin{aligned} \mathbf{n}\times(\mathbf{G}\times\mathbf{u}) &= (\mathbf{n}\cdot\mathbf{u})\mathbf{G}-(\mathbf{n}\cdot\mathbf{G})\mathbf{u}, \\ (\mathbf{G}\cdot\mathbf{u})\mathbf{n} &= \mathbf{G}\times(\mathbf{n}\times\mathbf{u})+(\mathbf{n}\cdot\mathbf{G})\mathbf{u}, \end{aligned} \tag{A-5}$$

so that

$$\iiint_V [\nabla \times (\mathbf{G} \times \mathbf{u}) + \nabla(\mathbf{G} \cdot \mathbf{u})] dV = \oiint_S [(\mathbf{n} \cdot \mathbf{u})\mathbf{G} + \mathbf{G} \times (\mathbf{n} \times \mathbf{u})] dS. \qquad \text{(A-6)}$$

One is then left with the expression

$$\oiint_S \{(\mathbf{n} \cdot \mathbf{u})\mathbf{G} + \mathbf{G} \times (\mathbf{n} \times \mathbf{u})\} dS = \iiint_V [2(\mathbf{u} \cdot \nabla)\mathbf{G} + \mathbf{G}(\nabla \cdot \mathbf{u}) - \mathbf{u}(\nabla \cdot \mathbf{G})$$
$$+ \mathbf{G} \times (\nabla \times \mathbf{u}) + \mathbf{u} \times (\nabla \times \mathbf{G})] dV. \qquad \text{(A-7)}$$

The divergence theorem may then be used to show that

$$\iiint_V (\mathbf{u} \cdot \nabla)\mathbf{G} dV = \oiint_S (\mathbf{n} \cdot \mathbf{u})\mathbf{G} dS - \iiint_V (\nabla \cdot \mathbf{u})\mathbf{G} dV, \qquad \text{(A-8)}$$

so that equation (A-7) becomes

$$\oiint_S [(\mathbf{n} \cdot \mathbf{u})\mathbf{G} - \mathbf{G} \times (\mathbf{n} \times \mathbf{u})] dS = \iiint_V \{\mathbf{G}(\nabla \cdot \mathbf{u}) + \mathbf{u}(\nabla \cdot \mathbf{G})$$
$$- \mathbf{G} \times (\nabla \times \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{G})\} dV. \qquad \text{(A-9)}$$

Now if one lets

$$\mathbf{G} = \frac{\mathbf{r}}{r(r + \varepsilon)^2} + \nabla H, \qquad \text{(A-10)}$$

where H is some function that is regular in the fluid domain, then, in the limit as ε approaches zero, one finds that

$$\iiint_V \mathbf{u}(\nabla \cdot \mathbf{G}) dV = -4\pi \mathbf{u},$$
$$\iiint_V \mathbf{u} \times (\nabla \times \mathbf{G}) dV = 0. \qquad \text{(A-11)}$$

Hence, equation (A-9) may be written as

$$4\pi \mathbf{u} = - \oint\!\!\oint_S \left[(\mathbf{n}\cdot\mathbf{u})\mathbf{G} - \mathbf{G}\times(\mathbf{n}\times\mathbf{u})\right]dS$$

$$+ \iiint_V \left[(\nabla\cdot\mathbf{u})\mathbf{G} - \mathbf{G}\times(\nabla\times\mathbf{u})\right]dV. \qquad \text{(A-12)}$$

In general, if the field point is taken to be in the domain, on the boundary of the domain, or in the complement of the domain, then the expression becomes

$$\left.\begin{matrix} 4\pi \text{ in D} \\ 2\pi \text{ on S} \\ 0 \text{ in } D^c \end{matrix}\right\} \mathbf{u} = - \oint\!\!\oint_S \left[(\mathbf{n}\cdot\mathbf{u})\mathbf{G} - \mathbf{G}\times(\mathbf{n}\times\mathbf{u})\right]dS$$

$$- \iiint_V \left[\mathbf{G}\times(\nabla\times\mathbf{u}) - (\nabla\cdot\mathbf{u})\mathbf{G}\right]dV. \qquad \text{(A-13)}$$

If **G** is taken to be the free-space Green's function

$$G = \frac{\mathbf{r}}{r^3}, \qquad \text{(A-14)}$$

then equations (A-12) and (A-13) become

$$4\pi \mathbf{u} = - \oint\!\!\oint_S \left[\frac{(\mathbf{n}\cdot\mathbf{u})\mathbf{r}}{r^3} - \frac{\mathbf{r}\times(\mathbf{n}\times\mathbf{u})}{r^3}\right]dS$$

$$- \iiint_V \left[\frac{\mathbf{r}\times(\nabla\times\mathbf{u})}{r^3} - \frac{(\nabla\cdot\mathbf{u})\mathbf{r}}{r^3}\right]dV, \qquad \text{(A-15)}$$

and

$$2\pi \mathbf{u} = - \oint\!\!\oint_S \left[\frac{(\mathbf{n}\cdot\mathbf{u})\mathbf{r}}{r^3} - \frac{\mathbf{r}\times(\mathbf{n}\times\mathbf{u})}{r^3}\right]dS$$

$$- \iiint_V \left[\frac{\mathbf{r}\times(\nabla\times\mathbf{u})}{r^3} - \frac{(\nabla\cdot\mathbf{u})\mathbf{r}}{r^3}\right]dV. \qquad \text{(A-16)}$$

APPENDIX B

DERIVATION OF SCALAR INTEGRAL IDENTITY

One may begin this derivation with Green's second identity (see reference 3):

$$\iiint\limits_V (\phi\nabla^2 G - G\nabla^2\phi)dV = \oiint\limits_S \left(\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n}\right)dS. \tag{B-1}$$

Now let

$$G = \frac{1}{(r + \varepsilon)} + H, \tag{B-2}$$

where H is some function that is regular in the fluid domain. Then it can be shown that

$$\lim_{\varepsilon \to 0} \iiint\limits_V \nabla^2\left(\frac{1}{r + \varepsilon}\right)dV = -4\pi, \tag{B-3}$$

so that, in the limit as ε approaches zero, one finds that

$$4\pi\phi = \oiint\limits_S \left(\frac{\partial \phi}{\partial n} G - \phi \frac{\partial G}{\partial n}\right)dS - \iiint\limits_V \nabla^2\phi G dV, \tag{B-4}$$

and, in general, if the field point is in the domain, on the boundary of the domain, or in the complement of the domain, this becomes

$$\left.\begin{matrix} 4\pi \text{ in D} \\ 2\pi \text{ on S} \\ 0 \text{ in } D^c \end{matrix}\right\} \phi = \oiint\limits_S \left(\frac{\partial \phi}{\partial n} G - \phi \frac{\partial G}{\partial n}\right)dS - \iiint\limits_V \nabla^2\phi G dV. \tag{B-5}$$

If G is taken to be the free-space Green's function

$$G = \frac{1}{r}, \tag{B-6}$$

then equations (B-4) and (B-5) take on the usual forms of Green's third identity:

$$4\pi\phi = \oiint_S \left[\frac{\partial\phi}{\partial n}\frac{1}{r} - \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] dS - \iiint_V \frac{\nabla^2\phi}{r}\, dV, \tag{B-7}$$

and

$$2\pi\phi = \oiint_S \left[\frac{\partial\phi}{\partial n}\frac{1}{r} - \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] dS - \iiint_V \frac{\nabla^2\phi}{r}\, dV. \tag{B-8}$$

APPENDIX C

PROOF OF WALL VORTICITY VECTOR'S TANGENCY

For any two orthogonal unit vectors tangent to the boundary t_i and s_i, the unit vector normal to the boundary may be expressed as

$$n_i = \varepsilon_{ijk} t_j s_k \, , \tag{C-1}$$

where ε_{ijk} is the alternating tensor. Since the vorticity is defined as (see Jeffreys (reference 5))

$$\omega_i = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} \, , \tag{C-2}$$

the normal component of the vorticity at the wall may be written as

$$\begin{aligned} n_i \omega_i &= (\varepsilon_{ijk} t_j s_k) \left(\varepsilon_{ilm} \frac{\partial u_m}{\partial x_l} \right) , \\ &= (\varepsilon_{ijk} \varepsilon_{ilm}) t_j s_k \frac{\partial u_m}{\partial x_l} \, , \\ &= (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}) t_j s_k \frac{\partial u_m}{\partial x_l} \, , \\ &= t_j s_k \left(\frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k} \right) , \\ &= s_k \left(t_j \frac{\partial u_k}{\partial x_j} \right) - t_j \left(s_k \frac{\partial u_j}{\partial x_k} \right) . \end{aligned} \tag{C-3}$$

Each of the last two terms represents a directional derivative tangent to the boundary. If one assumes that the boundary motion is rigid, then on the boundary

$$u_i = U_i + \varepsilon_{ijk} x_j \Omega_k \, , \tag{C-4}$$

where U_i is the translational motion of the boundary and Ω_i is its rotational motion about the origin of coordinates. With this representation one finds that

$$t_j \frac{\partial u_i}{\partial x_j} = \varepsilon_{ijk} t_j \Omega_k ,$$
$$= n_i(s_k \Omega_k) - s_i(n_k \Omega_k) , \quad \text{(C-5)}$$

and

$$s_j \frac{\partial u_i}{\partial x_j} = \varepsilon_{ijk} s_j \Omega_k ,$$
$$= t_i(n_k \Omega_k) - n_i(t_k \Omega_k) , \quad \text{(C-6)}$$

so that

$$n_i \omega_i = s_k(t_j \varepsilon_{kjm} \Omega_m) - t_j(s_k \varepsilon_{jkm} \Omega_m),$$
$$= 2\varepsilon_{kjm} s_k t_j \Omega_m ,$$
$$= -2n_m \Omega_m . \quad \text{(C-7)}$$

Thus, for rigid boundary motion, the normal component of the vorticity may be determined from the velocity boundary conditions.

The cross product of the normal and the vorticity at the wall may be determined in a similar manner. The cross product is defined as

$$\varepsilon_{ijk} n_j \omega_k . \quad \text{(C-8)}$$

If one employs equation (C-2), this expression becomes

$$\varepsilon_{ijk} n_j \omega_k = \varepsilon_{ijk} \varepsilon_{klm} n_j \frac{\partial u_m}{\partial x_l} ,$$
$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) n_j \frac{\partial u_m}{\partial x_l} ,$$
$$= n_j \left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \right) , \quad \text{(C-9)}$$

so that

$$\varepsilon_{ijk} n_j \omega_k = n_j\left(t_k \frac{\partial u_j}{\partial x_k} t_i + s_k \frac{\partial u_j}{\partial x_k} s_i + n_k \frac{\partial u_j}{\partial x_k} n_i\right) - n_j \frac{\partial u_i}{\partial x_j},$$

$$= n_j\left(t_k \frac{\partial u_j}{\partial x_k} t_i + s_k \frac{\partial u_j}{\partial x_k} s_i\right) - \left(n_j \frac{\partial u_i}{\partial x_j} - n_k n_j \frac{\partial u_k}{\partial x_j} n_i\right),$$

$$= n_j[(n_j\, s_k \Omega_k - s_j\, n_k \Omega_k) t_i + (t_j n_k \Omega_k - n_j t_k \Omega_k) s_i]$$

$$- \left(n_j \frac{\partial u_i}{\partial x_j} - n_k n_j \frac{\partial u_k}{\partial x_j} n_i\right),$$

$$= [(s_k \Omega_k) t_i - (t_k \Omega_k) s_i] - \left(n_j \frac{\partial u_i}{\partial x_j} - n_k n_j \frac{\partial u_k}{\partial x_j} n_i\right); \qquad \text{(C-10)}$$

hence, the components of the cross product are

$$t_i(\varepsilon_{ijk} n_j \omega_k) = s_k \Omega_k - t_i n_j \frac{\partial u_i}{\partial x_j},$$

$$s_i(\varepsilon_{ijk} n_j \omega_k) = -t_k \Omega_k - s_i n_j \frac{\partial u_i}{\partial x_j}, \qquad \text{(C-11)}$$

$$n_i(\varepsilon_{ijk} n_j \omega_k) = 0.$$

Since, for rigid boundary motion, the cross product of the boundary normal vector and the boundary vorticity contains derivatives of the velocity in the direction normal to the boundary, this cross product cannot be determined from the boundary conditions alone.

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